



TechTest2009

**Merit Scholarship Examination
in the Sciences and Mathematics
given on 4 April 2009**

sponsored by

***The Sierra Environmental Studies
Foundation***

Summary of rules and regulations for the exam:

1. Students will bring their own unmarked blue books to the exam; 3 to 5 blue books should suffice. On the cover of each blue book print your name, today's date '4 April 2009', the name of the exam 'TechTest2009', and the number/totalNumber of the blue book; e.g. the second of three blue books would be labeled 2/3.
2. Students will have four hours to take the test, 9am-1pm. Blue books not handed in by 1pm will not be graded.
3. There will be no passing or exchanging any kind of materials or information between test takers. If there is a problem, contact a proctor.
4. TI and Sanyo calculators may be used and the graphing types, TI 83 through TI-89, are expected.
5. Two, 8.5" X 11" 'cheat sheets' may be used under the following conditions: They are handwritten (both sides if desired) with your name clearly printed on each sheet.
6. No cell phones or communication devices of any kind may be used at any time during the test.
7. Students may leave the test to go to the restroom, one at a time.
8. Juniors will be allowed to come to the test and take it under the rules here stated, but will not be considered for scholarships. The eligibility requirements for Juniors remain the same, 3.0 min gpa, specified majors etc.
9. The determination and announcement of the winners is final. No re-grading or petitioning of any kind is allowed once the winners are determined.
10. All students will do and submit only their own work.
11. Recommendation: Read all the problems first and note their point values (Nine problems, 100 points total). Since partial credit will be given, make a plan on which order to solve each problem and to what point of completion so that you maximize your score in the time allowed.

Problem 1. [9 points] a) What is the product of the first five prime numbers? b) Why is the product of the first two or more prime numbers always even?

Problem 2. [9 points] Prove the following trigonometric identities.

$$\begin{aligned} \text{a) } & \sec^2\theta \csc^2\theta = \sec^2\theta + \csc^2\theta \\ \text{b) } & \csc x = \frac{1}{2} \left[\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} \right] \end{aligned}$$

Problem 3. [15 points] In error analysis the uncertainty δf in the output of a multi-variate model $f(\underline{x})$ is given by

$$\delta f(\underline{x})|_{\underline{x}_0} = \left[\sum_{i=1}^N \left(\left. \frac{\partial f}{\partial x_i} \delta x_i \right|_{\underline{x}_0} \right)^2 \right]^{\frac{1}{2}}, \quad \underline{x} = [x_1, x_2, \dots, x_N],$$

where \underline{x} is the vector notation of the set of independent variables in the model and the δx_i are their uncertainties. \underline{x}_0 denotes the ‘point’ in the variable space at which the model function f and its partial derivatives are evaluated. So the above formula for uncertainty holds at the point \underline{x}_0 .

Suppose you are to determine the acceleration of gravity g (cm/s^2) using the well-known pendulum experiment which gives the period T (s) of the pendulum of length l (cm) as

$$T = 2\pi \sqrt{\frac{l}{g}} \quad (\text{the model}).$$

In the experiment you measure $l = 90.52 \pm 0.1$ cm, and $T =$

1.910 ± 0.005 sec. Express your answer as $g \pm \delta g$ and evaluate the result in light of the usually accepted value of $g = 981$ cm/s^2 .

Problem 4. [10 points] The total tax burden is desired for a taxpayer who pays income taxes and buys something that has a sales tax. Let E be the dollars earned on which an income tax rate of T_I is levied. Let P be the price of something subject to a sales tax rate of T_S . Under this system of taxation how much does the taxpayer have to earn in order to complete the desired purchase – i.e. what is the total tax burden? Also, derive the formula for the incremental increase in earnings ΔE required for an incremental increase in the income tax rate ΔT_I . Use your results to answer both questions when $P = \$100$, $T_I = 20\%$, $T_S = 10\%$, and $\Delta T_I = 1\%$. (Hint: see problem #3)

Problem 5. [8 points] Conversion of quantities from one set of units to another is easy if you remember that conversion is nothing more than multiplying by a dimensionless unity expressed properly and treating all unit labels as algebraic variables. To convert centimeters to inches, we recall that 2.54 cm = 1 inch which allows us to express unity in two different ways –

$$\frac{2.54(\text{cm})}{1(\text{inch})} = \frac{1(\text{inch})}{2.54(\text{cm})} = 1 \text{ (pure numeric).}$$

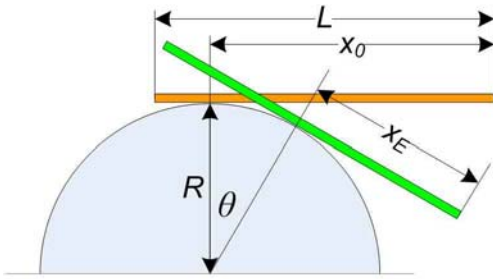
Therefore, to convert 7 cm into inches, we write

$$x(\text{inch}) = 7(\text{cm}) * 1 = 7(\text{cm}) * \left[\frac{1}{2.54} * \frac{\text{inch}}{\text{cm}} \right] = \frac{7}{2.54} * \text{inch} = 2.76 \text{ inches}$$

- What is the acceleration of gravity g in furlongs per fortnight squared? One furlong is one eighth of a mile, and a fortnight is two weeks. Take $1g = 32.174 \text{ ft/s}^2$.
- In automotive engineering the conversion of torque T , delivered by an engine rotating at a given RPM , to horsepower HP is a standard mechanics problem in physics. Derive the formula that computes horsepower in terms of torque expressed in $\text{ft} \cdot \text{lbs}$ and engine rotation in revolutions per minute. Recall that 1 horsepower = $550 \text{ ft} \cdot \text{lbs}/\text{sec}$, and that rotational power is $T\omega$ where ω is the angular velocity in radians/sec.

Problem 6. [15 points] Rain through a missing windshield. a) What amount of rain (liters/sec) comes through a rectangular hole, (c meters wide, b meters high) that is tilted at 45 deg from vertical, when the car moves horizontally at a speed of v_C (m/s) as the rain is falling straight down at v_R (m/s), and has a uniform density of ρ (l/m^3)? b) Suppose $c = 1.7 \text{ m}$, $b = 0.5 \text{ m}$, $v_R = 30 \text{ m/s}$, and $\rho = 0.05 \text{ l}/\text{m}^3$. If the car can reach a maximum speed of $v_{C,max} = 200 \text{ km/h}$, at what speed v_C^* m/s (using the star to indicate an extremizing value) in that range should it be driven so as to maximize the amount of rain \dot{I}_R^* l/s (using the dot symbol to indicate time rate of change) coming through the windshield hole? What is the value of \dot{I}_R^* l/s?

Problem 7. [12 points] Few people notice that it is easy to balance a stick on a ball, and fewer inquire why that is so. Just place the stick on the ball so that its approximate center point touches near the top of the ball, and the stick will automatically tilt in such a way that it comes to a point of equilibrium at an angle to the horizontal. Suppose you have a stick of constant linear density ρ (kg/m) that is L meters long. You also have a sphere (ball) of radius R (m) made of a non-deforming substance such that it prevents the stick from sliding along its surface. You place the stick on the top of the ball so that its original tangent point is x_0 from one end, and you slowly release the stick so that it comes to rest (equilibrium) at some angle θ from the vertical. Let $x_0 > L/2$ with no loss of generality. The figure shows the problem set up with the original and equilibrium positions of the stick in the 'plane of action'.



a) Derive the formula that expresses the angle θ and the length x_E to the tangent point at equilibrium in terms of the given parameters. b) Discuss how θ and x_E vary as x_0 approaches L . Remember that the stick cannot slide on the ball as long as $\theta < 90^\circ$. c) Compute the value of θ and x_E for $L = 1.5$ m, $R = 1.1$ m, $\rho = 3$ kg/m, $x_0 = 0.9$ m. Hint: Moments and torques are measured the same way.

Problem 8. [12 points] In an experiment, a biomass is known to go through three phases of growth. The fractional weight changes during each phase are given by f_1, f_2, f_3 such that in phase i the mass increases by $\Delta m_i = f_i m_{i-1}$ where m_{i-1} was the mass at the end of phase $i-1$. a) What is the total fractional weight gain F at the end of the experiment, i.e. at the end of phase three? b) Suppose F was found to be in error by an amount ΔF , and this error was isolated to a measurement error Δf_2 in f_2 . Express Δf_2 in terms of the given parameters and simplify your answer. c) In the experiment you measured $f_1 = 0.07, f_2 = 0.13, f_3 = -0.05$, and $\Delta F = -0.02$, what is Δf_2 ?

Problem 9. [10 points] A point (x_P, y_P) in the original (unprimed) cartesian coordinate system is mapped into a new (primed) cartesian coordinate system which is obtained by translation then rotation from the original system. a) If the origin of the primed system is translated by x_T and y_T in the unprimed system, and then rotated through the CCW angle θ , draw the appropriate figure and derive the relations for the coordinates in the primed system (x'_P, y'_P) in terms of the transformation variables and the original coordinates (x_P, y_P) . b) For $(x_P, y_P) = (2, 1), x_T = 5, y_T = 3, \theta = 10^\circ$, what are values for (x'_P, y'_P) ?